Modelling modal-split and trip length in a simultaneously discrete-continuous setting [1]

With an application for the city of Aarhus

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Abstract

Econometrics models for traffic behaviour as modal-split, car ownership, travel purpose, etc. is often based on the assumption that individual behaviour can be decomposed into a finite set of discrete choice. However, in some circumstances, for instance in modelling actual demand measured as trip length, this approach fails to fit reality. This paper present a simultaneous model set-up for the joint decision of modal-split and trip length being continuous. The model is estimated in a general maximum likelihood framework and is based on a stratified sample for Aarhus, the second largest city in Denmark.

Introduction

Most traffic behaviour can in a natural way be decomposed into discrete chooses. By applying stochastic utility theory it is possible to assign a certain amount of utility for every event relevant to the respondent in the choice set. The amount of utility is measured by the indirect conditional utility function. Based on this utility function the usual stochastic models for polytomous response, so as joint logit, nested logit and probit models can be implemented. These models work very well for at large number of problems including modal-split, car ownership and travel purpose. However, in investigating certain control parameters affecting monetary travel cost and level of service it's necessary to realise that especially travel demand measured by trip length is far from exogenous. A gasoline tax for instance will affect the pure modal-split but also the average trip length. This is so because respondents in the short run will reorganise their shopping pattern and leisure activity to save travel costs. The long run effect comes from the fact that the higher cost will actually affect the location of work and residents. By denying this fact one will automatically underestimate the effect of these control parameters.

To take account of the endogenous nature of the travel demand one have to formulate models both including modal-split and travel demand[2]. Taking this step it's first of all important to realise that even though the nature of modal-split is discrete the travel demand is only poorly described in a discrete content. To avoid this situation a pure continuous formulation of travel demand will be used. The model structure is inspired by Ben-Akiva and Watanatada 1981.

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[1] This paper is part of the methodically work underlying ALTRANS and was presented at Traffic days at AUC 1996. ALTRANS is a national research project financed by The National Transport Council, The National Environmental Research Institute and The Danish Environmental Protection Agency.

[2] In principle there are several endogenous components in the choice situation. Important ones to be neglected in the former paper is car ownership and purpose. However, if we only seek elasticities for variable travel costs, including both monetary costs and travel speed the above setup seems appropriate.

A discrete formulation

The conceptual issues and general understanding of the continuous model to be developed is improved by looking at the discrete counterpart. The well known joint logit for modal-split and destination has the following form

(1) $$P_t(m,d) = \frac{e^{V_{t,md}}}{\sum_{m' \in M_t} \sum_{d' \in D_t} e^{V_{t,m'd'}}} \quad m \in M_t \wedge d \in D_t \qquad \forall t = 1,\ldots,T \quad and \quad \bigcup_t D_t = M^*$$

Where the indirect conditional utility function is represented by $V_{t,md}$, include both exogenous variables and model parameters. Note that the choice sets M_t and D_t representing mode and destination choice respectively are specific at the individual level indicating that all respondents in general have different possibilities. The more precise these sets can be formulated on the individual level the more efficient the estimation will be. The set M^* is the entire geography. To make the argumentation against the joint logit model more precise let $A(\cdot)$ be an area operator. If we let $\bar{d}$ be the average area for zones in D_t , then

$for \;\; A(\bar{d}) \to 0$	*estimation impossible*
$for \;\; A(\bar{d}) \to M^*$	*Bad spatial description* $\to$ *bad model for travel demand*[3]

The first statement is related to pure numerical considerations. One thing being even more relevant is the data collection problem. For $\bar{d}$ being very small the time and money spend on collecting spatial data will be enormous. In practice however there is often no possibility to change the way data is collected, one simply has to adapt the quality and amount available. Ben-Akiva and Watanatada 1981 writes

" The discrete summation form cannot be used in actual applications when the number of spatial alternatives and individuals are large..."

The second statement refers to the problem of modelling trip length on the basis of regions with a ruff zonal partition. It's essential to remember that the precision of the choice of trip length in a discrete setting[4] is limited to the size of the zones.

Spatial choice represented by continuous functions

Motivated by the discussion above lets look at a 2-dimensional continuous counterpart of the discrete logit. This is done by thinking of the actual geography placed in an ordinary co-ordinate system letting the x and y-axis represent east-west and north-south direction respectively. The scale is in kilometre. The usual discrete choice d is then recognised as a co-ordinate $\{p,q\}$.
Define first a spatial choice function

$G_t\left(m,\{p,q\} \middle| \{x,y\}\right)$: The probability that respondent t located in $\{x,y\}$ chooses the spatial alternative located in $\{p,q\}$ by mode m . This probability is clearly affected by several things, the most important ones being

- Attraction in the destination.

[3] Of course the limit $A(d) \to M^*$ reduces equation (1) to a pure modal-split.

[4] This can be done by forcing the choice of trip length into disjunctive intervals.

- Trip generation in the origin area.
- Level of service between different locations and by different modes.
- Monetary travel costs.
- Socio-economic characteristic.

Therefore define the following

$M(\{p,q\})$: The density of attraction in the spatial location $\{p,q\}$.

$S_{t,m}$: Socio-economic characteristic.

The probability that respondent t chooses a spatial alternative located in zone j is then given by

(2) $$P_t\left(m, j \middle| \{x,y\}\right) = \iint_{\substack{zone\\ j}} G_k\left(m, \{p,q\} \middle| \{x,y\}\right) M\left(\{p,q\}\right) dpdq$$

By assuming the IIA-property and further letting $A(j) \rightarrow 0$ the continuous logit appears as the infinitesimal limit of the ordinary joint logit with probability

(3) $$P_t\left(m, dpdq \middle| M^*\right) = \frac{K_t\left(m, \{p,q\}\right) M\left(\{p,q\}\right) dpdq}{\sum_{m'} \iint_{\{p,q\} \in M^*} K_t\left(m', \{p,q\}\right) M\left(\{p,q\}\right) dpdq}$$

Here $dpdq$ denote a infinitesimal area and the spatial choice function $K_t(\cdot,\cdot)$ is now a function of mode and destination alone. In other words, we reject the dependence of zonal trip generation event though this could be included in a similar manner to attraction[5].

Application for the city of Aarhus

In order to understand applicability and how to implement the model above we shall look at a stratified TU-sample[6] for the city of Aarhus. The sample consist of 1008 records, each representing a trip in the area of Aarhus city. Only public transport and car as driver have been analysed as mode choice. In other words we have a binary choice situation neglecting cycling and car as passenger. This is far from satisfactory since cycling cannot be assumed to be equally competitive to car as driver and public transport. Look at figure 1. for a slightly compressed picture of the region.

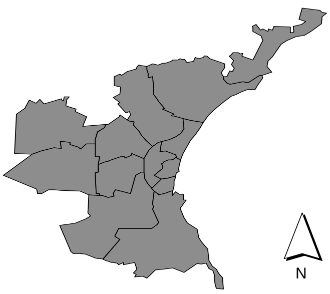

Figure 1- Aarhus partitioned in TU-zones

[5] Consult Ben-Akiva and Watanatada 1981 for further details.

[6] The TU-sample is result of an ongoing national data collection project financed by The Ministery of Transport.

The region is separated into 13 TU-zones in which aggregate attraction measures is available by combining the TU-database with databases from Denmark's Statistics.
The functional form of the utility function is supposed to be linear in parameters. More precisely let the function be of the following form, vectors being represented by bold.

(4) $$V_{t,md} = \mathbf{a}_m \mathbf{S}_{tm} + \mathbf{b}_m \mathbf{C}_{tmd} + b_d \gamma_d$$

$\mathbf{S}_{tm}$: {Constant, male, number of cars, distance to bus}

$\mathbf{C}_{tm}$: {Costs/income, travel time/trip length}

γ_d : {attraction density}

Formulating $\mathbf{S}_{tm}$ perfectly is difficult. This specification has been developed only as a primary form[7]. The series of $\mathbf{C}_{tm}$ is a function of several things such as, price of gasoline, tax deduction, age of car, cold starts, county[8] and more. It is important because scenarios typically originate from different specifications of these underlying factors[9]. Substituting this in the joint logit form gives

(5) $$P_t(m,d) = \frac{e^{\mathbf{a}_m \mathbf{S}_{tm}} \gamma_d e^{\mathbf{b}_m \mathbf{C}_{tmd}}}{\sum_{m'} e^{\mathbf{a}_{m'} \mathbf{S}_{tm'}} \sum_{d'} \gamma_{d'} e^{\mathbf{b}_{m'} \mathbf{C}_{tm'd'}}}$$

Now the remaining task is to formulate a continuous counterpart from the above model for the probability $P_t(m,d)$. This is done by transforming the ordinary rectangular co-ordinate system to one measured in polar co-ordinates. In other words let $d = \{x, y\} \rightarrow \{L, \theta\}$. By this we get

- $C_{t,md} \rightarrow C_{t,m}(L,\theta)$
- $\gamma_d \rightarrow \gamma(L,\theta)$

The smart thing about this is that we can treat L as the endogenous travel demand. This is done by shifting the centre of the co-ordinate system for every t , so that the new centre is the origin from where respondent t travels. The new city geometry is illustrated in figure 2. Using this trick the new model for the probability $P_t(m, dLd\theta)$ is given as

(6) $$\begin{aligned} P_t(m, dLd\theta) &= G_t(m,\{L,\theta\})dLd\theta \\ &= \frac{e^{\mathbf{a}_m \mathbf{S}_{tm}} \gamma(L,\theta) e^{\mathbf{b}_m \mathbf{C}_{tm}(L,\theta)} dLd\theta}{\sum_{m} e^{\mathbf{a}_{m'} \mathbf{S}_{tm'}} \int_0^B \int_0^{\frac{v}{360}2\pi} \gamma(L,\theta) e^{\mathbf{b}_{m'} \mathbf{C}_{tm'}(L,\theta)} \left| \frac{\partial(x,y)}{\partial(L,\theta)} \right| dLd\theta} \end{aligned}$$

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[7] More insight about variables affecting the modal split using TU-data can be gained from ALTRANS *internal working paper* no.2 1996, Nogle foreløbige resultater i ALTRANS.
[8] Taking account of different kilometer costs for public transport in different counties.
[9] For further details consul ALTRANS *internal working paper* no.1 1996, Konstruktion af rejseomkostninger for bil og kollektiv.

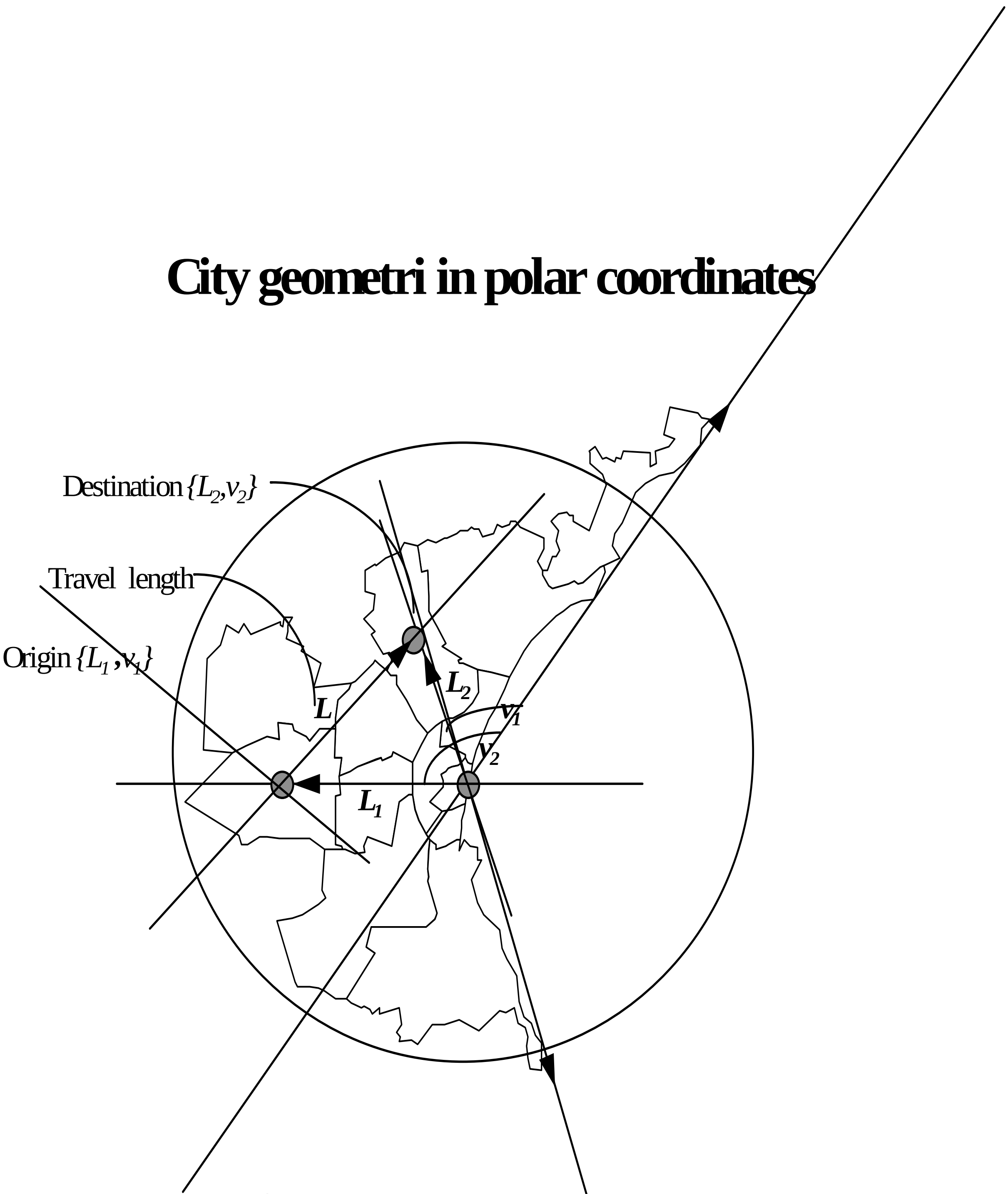

Figure 2 - Polar city geometry for Aarhus

Where v constrains the possible geography from the bay of Aarhus. Note that when $dLd\theta$ approaches 0 then $P_t(m, dLd\theta)$ approaches $P_t(m, \{L, \theta\})$.

Considerations of the $\gamma(L, \theta)$ - form

To illustrate how the form of the spatial component $\gamma(L, \theta)$ could be approximated by a simple function look at a spline-approximation of a discrete attraction density defined by employed in retail per square meters. This is illustrated in figure 3. below

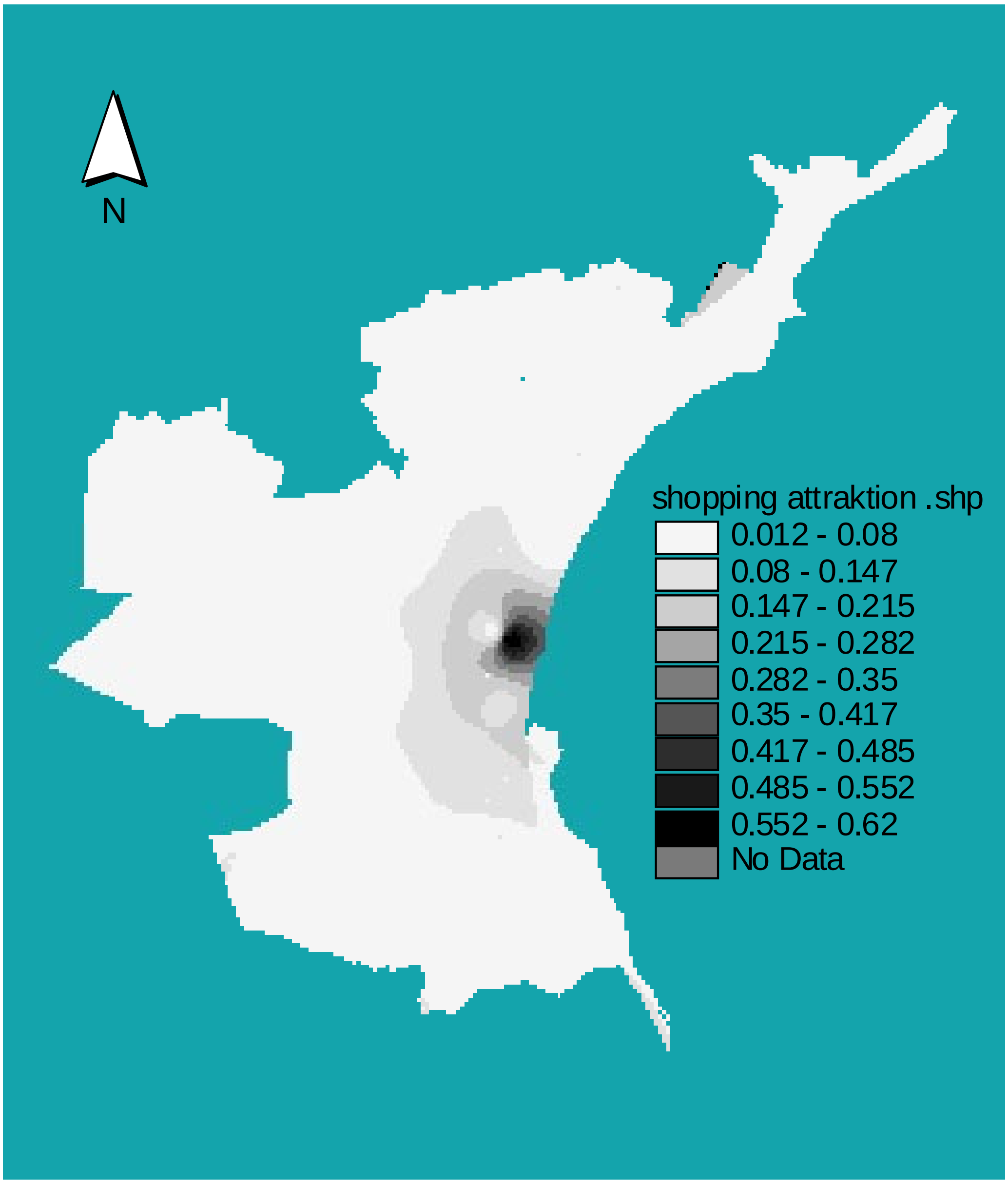

Figure 3 - Spline approximation of attraction density

The figure is developed in ARC/VIEW 3.0 and the main thing to notice is the systematic decreasing attraction density as a function of distance from the city-centre. It is more convenient to look at a vertical cut of the above surface. This is plotted below in figure 4.

The exponential function illustrated in figure 4. seems to be a reasonably approximation. The functional form is given by

(7) $\gamma(L,\theta) \sim 0.4460 * 0.999534^{L}$

Where L is the distance from the city centre.

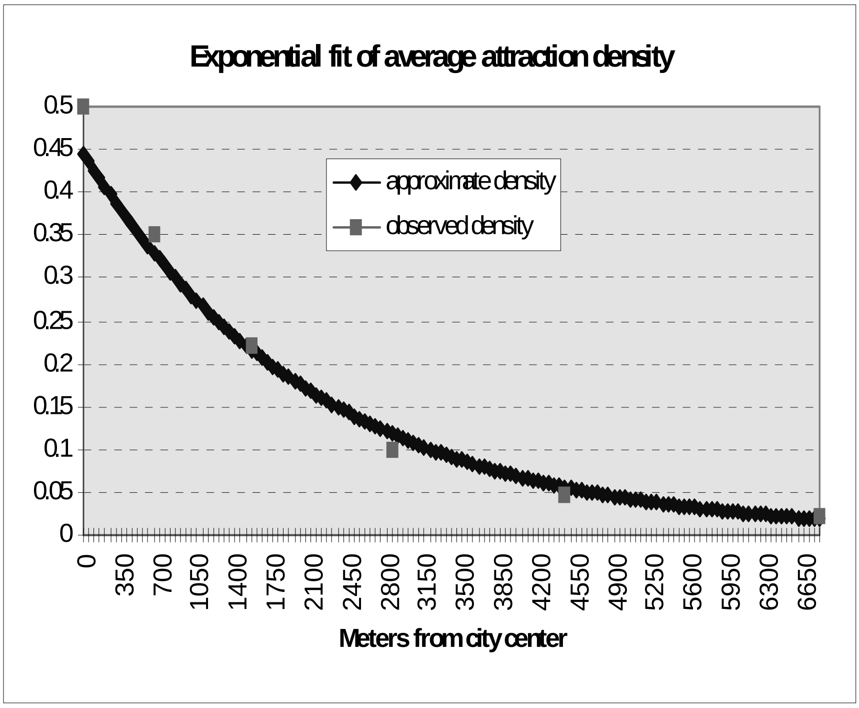

Figure 4

Unfortunately even simple functional forms like the one above is quite difficult to handle due to the integral in the denominator of (6). Firstly of all there is no closed form expression so it's necessary to evaluate the double integral by numerical integration. Secondly the domain for the integration will change in a quite complex manner. These two things means that function call's becomes extremely expensive and this connected to the fact that an estimation needs at least 20 functions call's makes the numerical burden quite heavy[10].To overcome this problem we chooses the gamma function even more simple as

(8) $$\gamma(L,\theta)=\gamma$$

Using the above featureless plane for the attraction density have some nice computable implications as we shall see below. The simplification have on the other hand some implications related to model structure and level of analysis. First of all the featureless plane only allow a relative geographical description of travel behaviour, meaning that the spatial pattern for trips defined by origin co-ordinates and destination co-ordinates is not modelled. So if we're interested in modelling locations of work and residence we have to seek a more general form of $\gamma(L,\theta)$ or simply turn back to the good old discrete formulation. Note here that even though we have a mathematical correct description of the attraction density we cannot allocate individuals more precise than the observation allow. For instance, in the case of Aarhus the spatial allocation of trips (origins and destinations) is measured at a TU-zonal level. Another thing to note is that the above simple gamma function does not prevent us from using discrete attraction measures as explanatory variables[11].
In summarising the above it's fair to say that the featureless continuous logit seems to be a good short run model for travel behaviour conditional on attraction and generation.

Model Estimation

With the above utility form and specification of attraction density the denominator integral can be evaluated exact. We have that

[10] Note that for every call of the likelihood function we need to evaluate N integrals. In our small example we will need to evaluate the denominator integral at least 20000 times.
[11] The exogenious treatment of attraction is also the case in the joint-logit in (1).

(9)
$$\int_0^B \int_0^{\frac{v}{360}2\pi} \gamma e^{\mathbf{b}_m \mathbf{C}_{tm}} dLd\theta = \gamma \int_0^B \int_0^{\frac{v}{360}2\pi} Le^{b_{m1}L + b_{m2}L} dLd\theta$$
$$= 2\pi \frac{v}{360} \gamma \left[\frac{1 + e^{B(b_{m1}+b_{m2})}\left(B\left(b_{m1}+b_{m2}\right)-1\right)}{\left(b_{m1}+b_{m2}\right)^2} \right]$$

It is now possible to state the probability for the combined choice of modal-split and travel demand assuming a featureless plane as

(10)
$$P_t\left(m,\{L,\theta\}\right) = \frac{360}{2\pi v} \frac{e^{\mathbf{a}_m \mathbf{S}_{tm}} e^{\mathbf{b}_m \mathbf{C}_{tm}(L)}}{\sum_{m' \in M_t} e^{\mathbf{a}_{m'} \mathbf{S}_{tm'}} \left[\frac{1 + e^{(b_{m'1}+b_{m'2})B}\left(B(b_{m'1}+b_{m'2})-1\right)}{(b_{m'1}+b_{m'2})^2} \right]}$$

The smart thing about the featureless attraction plane from a numerical point of view is that for all T denominator integrals can make an artificial co-ordinate shift assuming that the centre is exactly where respondent t begin his trip. Since all variables are relative to trip length the above form will do.
Unfortunately this model is not covered by standard software packages. However, the model can be programmed in a SAS/IML environment. Firstly specify the log-likelihood function

(11)
$$\lambda\left(\mathbf{a}_m, \mathbf{b}_m\right) = \sum_{t'}^{T} \sum_{m' \in M_t} y_{tm'} \log\left(P_{t'}(m', dLd\theta)\right)$$
$$= T \log\left(\frac{360}{2\pi v}\right) +$$
$$\sum_{t'}^{T} \sum_{m' \in M_t} y_{tm} \left(\left(\mathbf{a}_{m'} \mathbf{S}_{t,m'} + b_{m'1}L + b_{m'2}L\right) - \log\left(\sum_{m' \in M} e^{\mathbf{a}_{m'} \mathbf{S}_{tm'}} \left[\frac{1 + e^{(b_{m'1}+b_{m'2})B}(B(b_{m'1}+b_{m'2})-1)}{(b_{m'1}+b_{m'2})^2} \right] \right) \right)$$

$\mathbf{a}_m$ and $\mathbf{b}_m$ being parameter vectors associated to individual specific and mode specific variables. Further

$$y_{tm} = \begin{cases} 1 & if \quad individual \quad t \quad chose \quad m \\ 0 & else \end{cases}$$

The maximum likelihood estimates[12] is now given by

[12] Since the above likelihood function is not globally concave we must be careful to ensure that the parameters is asymptotically identified, meaning that $\lambda(\bar{\varphi}) > \lambda(\varphi) \quad \forall \varphi \in \Theta$ where φ a set of parameters and Θ the complete range of parameters. This is done more or less automatically in SAS/IML by choosing several numerical optimization routines including conjugate gradient optimization. This routine generate a cycle of conjugated search directions. For more about these tecnical issues consult SAS/IML User guide and SAS/IML Changes and Enchancements.

	Parameter	Gradient	Asymptotic P-Value
constant	-4.430233	0.0000253	5.11E-10
number of cars	4.835476	-0.0000131	1.37E-10
male	3.335879	0.0000246	4.83E-10
distance from bus	0.566504	0.0000195	3.03E-10
time for car driver	-0.149246	0.0000797	5.07E-09
time for public transport	-0.02492	-0.000268	5.73E-08
costs for car driver	-0.122078	0.0000816	5.31E-09
Costs for Public transport	-0.231764	-0.000223	3.97E-08

All parameters are significant[13] and all signs seem to be right. However, as the model is non-linear in the exogenous variables the marginal effects have to be investigated using elasticities.

Elasticities and marginal effects

The main purpose of the model is to determine the sensibility of demand after public transport due to changes in travel time and monetary costs. The first thing to calculate is the elasticities related to the pure modal split, measuring the re-allocation potential from a environmental point of view. The computation of the elasticities is based on the form for the individual elasticities given by

(12) $$E_{x_{tjk}}^{P_t(i)} = \frac{\partial P_t(i)}{\partial x_{tjk}} \frac{x_{tjk}}{P_t(i)}$$

However we're in no way interested in elasticities on an individual basis but in aggregate elasticities being defined as

(13) $$E_{x_{jk}}^{\overline{P}(i)} = \frac{\sum_{n \in N_i} P_n(i) E_{x_{jnk}}^{P_n(i)}}{\sum_{n \in N_i} P_n(i)}$$

Measuring elasticities on the aggregate modal-split probability due to changes in x_{jk}. Below is the direct elasticities ($i = j$) outlined[14].

	Time for car driver	Time for public transport	Cost for car driver	Cost for public transport
elasticities for modal-split probabilities	-0.1826	-0.4069	-0.0299	-0.4486

Even though the elasticities all have the right sign the range of the size estimated can be discussed. However, when looking at a stratified sample it's necessary to be careful when interpreting these numbers. For more serious analysis much more effort has to be put into the specification of the conditional utility function.

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[13] Since the covariance matrix is quite hard to program exact we use the following asymptotical estimator $\left[\left(\frac{\partial\lambda(\varphi)}{\partial\varphi}\right)^T\left(\frac{\partial\lambda(\varphi)}{\partial\varphi}\right)\right]^{-1}$ for the information matrix. Since SAS rapports $\frac{\partial\lambda(\varphi)}{\partial\varphi}$ this estimator becomes cheap.

[14] A little tecnical hint that might save some time in front of the computer is: Evaluate the differential kvotient numerically when calculating the elsaticities.

The next effect to investigate is the effect on the trip length. In other words, how much costs and travel time affect the average trip length. This can be done in a quite simple manner. First of all we're finding the marginal function for trip length given by

$$(14) \qquad f_t(L|m) = \frac{(b_{m1}+b_{m2})Le^{(b_{m1}+b_{m2})L}}{(1+e^{(b_{m1}+b_{m2})B}((b_{m1}+b_{m2})B-1))}$$

Then recognise that

$$(15) \qquad \begin{aligned} \lim_{B\to\infty} f_t(L|m) &= (b_{m1}+b_{m2})^2\, Le^{L(b_{m1}+b_{m2})} \\ &= \Gamma(2, b_{m1}+b_{m2}) \end{aligned}$$

For $B < \infty$ the above gamma-distribution becomes truncated. For reasonably large B (≈ 50 km in this model) the truncated distribution becomes a very good approximation to the asymptotic gamma function. This indicates that the mean of the truncated distribution in practice will approximate the mean of the above gamma function given by $\frac{2}{b_{m1}+b_{m2}}$. To illustrate lets assume that travel speed for public transport is changed. Then two sets of parameters occurs. Let these be given by $\{\tilde{b}_{m1}, \tilde{b}_{m2}\}$ and $\{\hat{b}_{m1}, \hat{b}_{m2}\}$ the first one being before change and the second being after. Now the change in percentage of average trip length for mode m is given by

$$(16) \qquad \left(1 - \frac{\tilde{b}_{m1}+\tilde{b}_{m2}}{\hat{b}_{m1}+\hat{b}_{m2}}\right)\cdot 100$$

One of the main advances of this model seems to be the quite precise elasticities for monetary costs and travel speed as we are modelling directly on trip length. This is particularly important when we want to investigate the effect of service for public transport measured as differences in travel time.
To validate the model further[15] look at some results derived from the above trip distribution function.

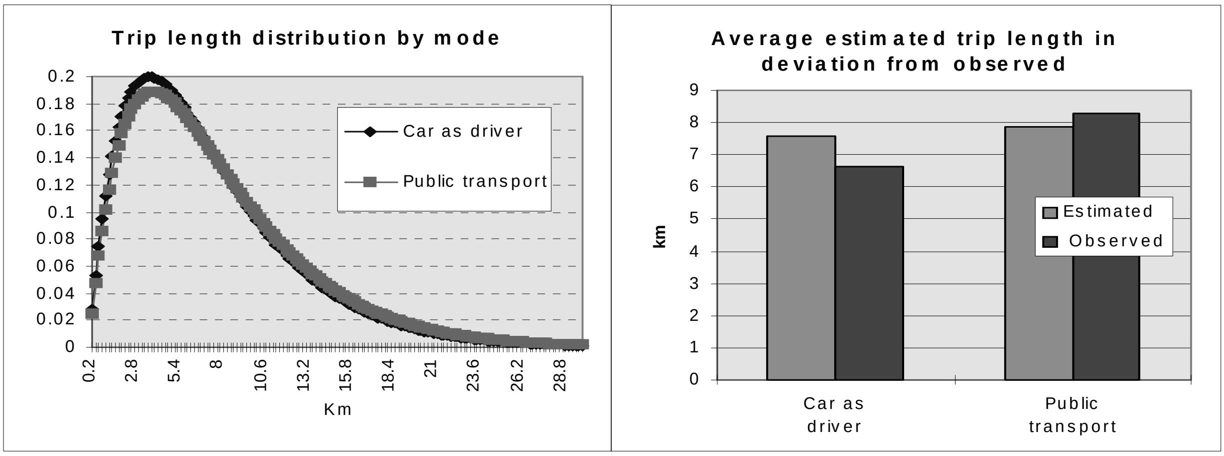

Figure 5

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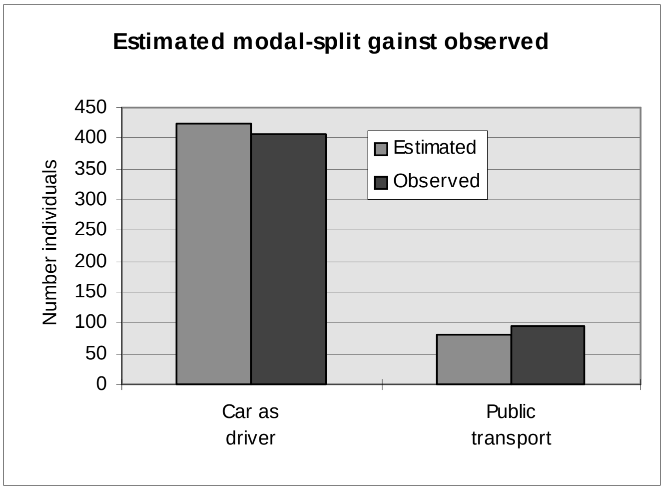
[15] For the pure modal-split model the IIA-property has been tested using a LR-test and it is easily accepted. However it's not evident how to test the infinitesimal IIA used to construct equation (6). Moreover note should be taken that it is impossible to test the precise gumbel assumption underlying the logit model as we're dealing with a latent variable situation not knowing the true utility function.

As mentioned the above trip length distribution in figure 5 is actually a truncated gamma function. Ben-Akiva and Watanatada 1981 point out that this form has actually been observed in several urban areas.
Another interesting marginal probability is the probability measuring the modal-split share. By some algebra one can find that

(17) $$P_t(m) = \frac{e^{\mathbf{a}_m \mathbf{s}_{tm}}[1+e^{(b_{m1}+b_{m2})B}((b_{m1'}+b_{m2})B-1)](b_{m1}+b_{m2})^{-2}}{\sum_{m'} e^{\mathbf{a}_{m'} \mathbf{s}_{tm'}}(1+e^{(b_{m'1}+b_{m'2})B}((b_{m'1}+b_{m'2})B-1))(b_{m'1}+b_{m'2})^{-2}}$$



As shown there seems to be a good degree of accordance between the observed and estimated modal-split. Also the estimated average trip length seems to be reasonably good.

Conclusion

In the short run it’s reasonably to assume fixed locations of working places and residents. By using a simple featureless plane of attraction it is possible to formulate a combined model for trip length and modal-split. Since we are modelling directly on the observed travel length, it seems likely that elasticities for service and travel costs will differ from those obtained from discrete models. This might result in more precise knowledge about marginal effects of public service measured as time-in-vehicle, terminal-time and waiting time being one of the main tasks in ALTRANS.
The model in this paper is only the first attempt to seek to test technical issues and possibilities. However, the model turns out to be quite successful for our purpose and it will enter as part of the ALTRANS project.

The model estimated in this paper uses an extremely simple functional form for the attraction density. In order to implement more complex functions numerical mathematics have to be developed. In an idealised form $\gamma(L,\theta)$ could be a parametric spline function. One thing that might point in a positive direction for further development is that the numerical problem can be formulated quite precisely leaving some hope, that even middle sized problems could be solved this way. However, the numerical motivation for choosing the continuous logit seems to disappear when choosing more complex forms of $\gamma(L,\theta)$.

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